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Non-isolated types in stable theories

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Abstract

We introduce notions of strong and eventual strong non-isolation for types in countable, stable theories. For T superstable or small stable we prove a dichotomy theorem: a regular type over a finite domain is either eventually strongly non-isolated or is non-orthogonal to a NENI type (in T^{eq}). As an application we obtain the upper bound for Lascar's rank of a superstable theory which is one-based or trivial, and has fewer than $2^{\aleph_0}$ non-isomorphic countable models.

Keywords: Superstable theory; Regular type; Small theory; Strongly non-isolated type; Fundamental order; Lascar's rank

The proof of Vaught's Conjecture for $\aleph_0$-stable theories, see [16], was enabled by understanding non-isolation properties of regular types in more detail. In $\aleph_0$-stable theories every regular type is non-orthogonal to a strongly regular type; and for strongly regular types we have a strong dichotomy: a strongly regular type is either ENI or NENI, depending on whether it is eventually (i.e. its non-forking extension over a finitely extended domain) non-isolated or not. NENI types have a strong isolation property: they are not just isolated, but eventually stay isolated as well. On the other hand, a non-isolated strongly regular type is not just non-isolated, but is also almost-orthogonal to any isolated type over the same or slightly larger domain, indicating that ENI types have (eventually) a strong non-isolation property. The dichotomy proved to be crucial in determining possible dimensions of regular types.

In this article we establish a similar dichotomy for regular types assuming that the underlying theory is either countable and superstable or small stable. We introduce the notion of strong non-isolation for types which, if the theory is assumed to be small, is equivalent to almost-orthogonality to all isolated types over the same or slightly larger domain. We prove that a regular type is either eventually strongly non-isolated (ESN) or is non-orthogonal to a NENI type. Hence, in the $\aleph_0$-stable case our ESN–NENI dichotomy coincides with ENI–NENI dichotomy.

Throughout the paper fix a countable, complete, stable theory T and its (infinite) monster model $\mathcal{M}$. The main result is:

Theorem 1 (T *superstable or small stable*). *Let* $p \in S(\emptyset)$ *be a regular type. The following conditions are equivalent:*

(A) *p is eventually strongly non-isolated;*

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(B) *for all finite B and all stationarizations p' of p there is a model $M \supseteq B$ such that $dim(q, M)$ is finite for all regular, stationary types $q \not\perp p'$ whose domain is a finite subset of M;*

(C) *p is orthogonal to all NENI-types whose domain is a finite subset of $\mathcal{M}^{eq}$.*

(A)$\Leftrightarrow$(C) in the above theorem gives the described dichotomy; it suggests that (in the case of stationary, regular types) the eventual strong non-isolation is a non-isolation property of the whole $\not\perp$-class rather than the property of a sole type. In this sense condition (B) in the theorem is an omitting-types theorem for the whole class: given a regular, stationary type $p \in S(\emptyset)$ we cannot, in general, omit all the types from the $\not\perp$-class of p, even when p is strongly regular and non-isolated. The best we can do is to make all dimensions of types from the class finite, as described by (B).

In [4] it is shown that an $\aleph_0$-categorical, $\aleph_0$-stable theory must have finite U-rank; in other words, if the underlying theory has infinite U-rank then there must exist a non-isolated type over $\emptyset$. The original proof used one-basedness of the theory. We use a similar argument to prove that if a type is one-based or trivial, and has limit-ordinal U-rank, then it must be eventually strongly non-isolated. This is used later in finding the upper bound for Lascar's rank of a superstable theory with few countable models which is one-based or trivial.

The paper is organized as follows. In Section 1, in order to approximate NENI types, we introduce internally isolated types. Roughly speaking, those are types p such that forking is a definable relation on $p(\mathcal{M})$. We prove that in the case of regular types over a finite domain internal isolation is preserved under non-orthogonality. In Section 2 we introduce strongly and eventually strongly non-isolated types and establish a few basic facts about them. Sections 3 and 4 are devoted to proofs of the superstable and the small, stable case of the theorem, respectively. In Section 5 we prove that ω^ω is the strict upper bound for Lascar's rank of a superstable, trivial or one-based, theory with few countable models.

The present article generalizes some results from my Ph.D. Thesis [17]. Theorem 1 was originally proved there for small superstable theories. Later on I realized that some of the topological methods developed by Newelski around the notion of meager forking (see [11–15]) may be used to replace my arguments involving the existence of ordinal-valued CB-rank by others which use only the Baire Category Theorem; as a result I proved the theorem in the countable superstable case. For the small stable case the proof was adapted by using again a result of Newelski's (Theorem 1.5 from [14]). Section 5 is completely contained in my Ph.D. Thesis; the only difference is that the proofs here are corrected.

Notation

We assume some basic knowledge of stability theory, as can be found in [1] and [10], and some knowledge of p-simplicity, as can be found in [7]. By $S_n(A)$ ($S_n^*(A)$) we shall denote the set of all complete n-types (strong n-types) over A. If p is an n-type over A, possibly incomplete, then by $[p]_A$ ($[p]_A^*$) we shall denote the set of all complete types (strong types) from $S_n(A)$ ($S_n^*(A)$) containing p. If $A = \emptyset$ then we simply write $[p]$ instead of $[p]_\emptyset$. Note that $[p]_A$ is a closed subset of $S_n(A)$, while $[p]_A^*$ is a closed subset of $S_n^*(A)$. If $p, q \in S(A)$ are stationary then their product, denoted by $p \otimes q$, is the type over A of any pair of realizations of p and q which are independent over A. Similarly we define the product $p_1 \otimes p_2 \otimes \cdots \otimes p_n$ and the power p^n of stationary types (or strong types).

We shall use Newelski's notion of traces; the trace of $\phi(\bar{x}, \bar{b})$ over A is defined by:

$$Tr_A(\phi(\bar{x}, \bar{b})) = \{r \in S(A) \mid r(\bar{x}) \text{ is consistent with } \phi(\bar{x}, \bar{b})\}.$$

$Tr_A(\phi(\bar{x}, \bar{b}))$ is a closed subset of $S(A)$; for more see [12]. If $A = \emptyset$ then we write $Tr(\phi(\bar{x}, \bar{b}))$.

If $p = tp(a/A)$, E is an A-definable equivalence relation and $e \in \mathcal{C}^{eq}$ is the name of the E-class of a (i.e. $e = a/E$) then by p/E we denote $tp(e)$. If $p, q \in S(A)$ then by $p \in dcl(q)$ we mean that for some (any) $a \models q$ there exists $b \models p$ such that $a \in dcl(b)$; similarly for $p \in acl(q)$.

We shall also use Hrushovski's quantifier: if $p \in S^*(A)$ and $\phi(x, y)$ is over A then $(d_p x)\phi(x, y)$ denotes a formula (over $acl^{eq}(A)$) which defines ϕ in p. If $p \in S(A)$ is stationary then $(d_p x)\phi(x, y)$ is over A.

If $p \in S(A)$ then sometimes we write $p(x)$ to distinguish the variable(s) used; if $p(x) = tp(b/A)$ then we also write $p = tp_x(b/A)$. $\phi(x; \bar{y})$ (over $\emptyset$) is represented in $p(x)$ if $\phi(x, \bar{a}) \in p$ for some $\bar{a} \in A$. The bound of p is:

$$bnd(p(x)) = \{\phi(x; \bar{y}) \mid \phi(x; \bar{y}) \text{ represented in every extension } q \in S(\mathcal{M}) \text{ of } p\}.$$

If $B \subseteq A$ then $bnd_B(p)$ is the bound of p in the language expanded by constants from B.

If $q \in S(B)$ and $A \subseteq B$ then q does not fork over A iff $bnd(q) = bnd(q|A)$. Even stronger: q forks over A iff there is $\phi(x; \bar{y}) \notin bnd(q|A)$ which is represented in q. We sometimes refer to such ϕ as a witness of forking of q.

For any ϕ and $p \in S(A)$ $\phi(x; \bar{y}) \in bnd(p)$ iff there is n and $\delta(v_1, v_2, \dots, v_n)$ over A such that:

$$\models (\exists \bar{v})\delta(\bar{v}) \wedge (\forall \bar{v}) \left(\delta(\bar{v}) \Rightarrow \bigvee_{1 \le i \le n} \phi(x, v_i) \right) \in p.$$

Thus $\{p \in S(A) \mid \phi(x; \bar{y}) \in bnd(p(x))\}$ is open in $S(A)$, and:

$$E_\phi = \{p \in S(A) \mid \phi(x; \bar{y}) \notin bnd(p(x))\} \text{ is closed in } S(A).$$

Types $p \in S(A)$ and $q \in S(B)$ (not necessarily stationary) are almost orthogonal, or $p \overset{a}{\perp} q$, if whenever $tp(\bar{a}/AB)$ is a non-forking extension of p and $tp(\bar{b}/AB)$ is a non-forking extension of q then $\bar{a} \mathop{\smile\!\!\!\!\!\!\;\mid\;} \bar{b}(AB)$. The opposite is denoted by $p \overset{a}{\not\perp} q$. p and q are orthogonal, or $p \perp q$, if whenever $AB \subseteq C$ and $tp(\bar{a}/C)$ is a non-forking extension of p and $tp(\bar{b}/C)$ is a non-forking extension of q then $\bar{a} \mathop{\smile\!\!\!\!\!\!\;\mid\;} \bar{b}(C)$.

p is regular if whenever q is a forking extension of p then $p \perp q$. $p \in S(A)$ is strongly regular if it is stationary and there is $\phi \in p$ such that: whenever $\phi \in q \in S(AB)$ then either $p \perp q$ or q is a non-forking extension of p.

Note that we allow regular types to be non-stationary while strongly regular types are assumed to be necessarily stationary.

1. Internally isolated types

Definition 1. A type $p \in S(A)$ is **internally isolated** if for every $n \in \mathbb{N}$ there exists a formula $\phi_n(x_1, x_2, \dots, x_n)$ over A such that for all $q \in [p]^*_A$ we have:

$$(q(x_1) \wedge q(x_2) \wedge \cdots \wedge q(x_n) \wedge \phi_n(x_1, x_2, \dots, x_n)) \Leftrightarrow q^n(x_1, x_2, \dots, x_n).$$

p is internally non-isolated if it is not internally isolated.

Example 1. Let $p \in S(\emptyset)$ be regular and such that the dependence relation is locally finite on $p(\mathcal{M})$, i.e. $\{x \in p(\mathcal{M}) \mid x \not\mathop{\smile\!\!\!\!\!\!\;\mid\;} A\}$ is finite whenever $A \subset p(\mathcal{M})$ is finite. We leave to the reader to verify that p is internally isolated.

Example 2. Let $p \in S_1(\emptyset)$ be the generic type of $\mathbb{Z}_2^{\aleph_0}$. Note that p is internally isolated, for example:

$$p^2(x, y) \text{ is equivalent to } p(x) \wedge p(y) \wedge x \neq y.$$

However, if $I \models p^\omega$ then $p|I$ is not internally isolated: if $a, b \models p|I$ then $ab \models (p|I)^2$ iff $a \pm b \notin dcl(I)$, and the latter cannot be described by a first-order formula.

The previous example shows that internal isolation is, in general, not invariant under parallelism. It is also not invariant under $\not\perp$ of regular types. However, we show in this section that it is the case if we assume that the underlying theory is superstable or small stable and consider only types over finite domains.

From now until the end of the section assume that T is superstable or small stable (although countability of T is assumed throughout the paper, in the superstable case of this section it is not essentially used). The following fact, due to Newelski, is the main consequence of the assumption used below:

Lemma 1.1. *Let $p \in S(\emptyset)$ have finite weight and let A be finite.*

(a) *If p is stationary then for all large enough integers n there is $\bar{b} \models p^n$ such that $p|\bar{b} \vdash p|A$.*

(b) *If $a_1, a_2, \dots$ is an infinite Morley sequence in p then $a_k \mathop{\smile\!\!\!\!\!\!\;\mid\;} A$ for some k.*

Proof. (a) In the superstable case, pick n large enough and $\bar{b} \models p^n$ such that $U(A/\bar{b})$ is minimal possible. In the small, stable case this is a special case of Theorem 1.5 from [14].

(b) Follows from (a) by calculating weights. $\square$

Recall that $p \in S(A)$ is **eni**, or eventually non-isolated, iff there is a finite set B and a non-isolated, non-forking extension of p in $S(AB)$. p is **ENI** if it is strongly regular and eni. p is **NENI** if it is strongly regular and is not eni (this slightly differs from the original definition from [16] in that we allow a NENI type to have infinite domain).

We have introduced internally isolated types in order to approximate NENI types; this is described by the next lemma.

Lemma 1.2. *A regular type $p \in S(\emptyset)$ is NENI if and only if it is isolated, internally isolated and stationary.*

Proof. $\Rightarrow$ is left to the reader; we prove $\Leftarrow$. For assume that p is isolated, internally isolated and stationary. Note that it follows that p^n is isolated for all n. Let A be finite and let $q \in S(A)$ be the non-forking extension of p. We shall show that q is isolated. By Lemma 1.1 there are n and $\bar{b} \models p^n$ such that $p|\bar{b} \vdash p|A$. p^{n+1} is isolated, so let $\psi(x_1, x_2, \ldots, x_{n+1})$ be a formula over $\emptyset$ isolating it. Let $r = stp(\bar{b}/A)$. It is straightforward to check that the following formula isolates q:

$$(d_r x_1 x_2 \ldots x_n)\psi(x_1, x_2, \ldots, x_n, x). \quad \Box$$

Lemma 1.3. *Suppose $p \in S(\emptyset)$ has finite weight.*

(a) *If there is $q \in [p]^*$ such that for all n there is a formula $\varphi_n(x_1, x_2, \ldots, x_n)$ over $acl^{eq}(\emptyset)$ such that*

$$(q(x_1) \wedge q(x_2) \wedge \cdots \wedge q(x_n) \wedge \varphi_n(x_1, x_2, \ldots, x_n)) \Leftrightarrow q^n(x_1, x_2, \ldots, x_n)$$

then p is internally isolated.

(b) *p is internally isolated if and only if some extension of p to $acl^{eq}(\emptyset)$ is internally isolated, if and only if all extensions of p to $acl^{eq}(\emptyset)$ are internally isolated.*

(c) *If B is finite and $q \in S(B)$ is a non-forking extension of p then p is internally isolated if and only if q is so.*

Proof. (a) Suppose that $q \in [p]^*$, $e \in acl^{eq}(\emptyset)$ and $\varphi_n(x_1, x_2, \ldots, x_n, e)$ satisfy:

$$(q(x_1) \wedge q(x_2) \wedge \cdots \wedge q(x_n) \wedge \varphi_n(x_1, x_2, \ldots, x_n, e)) \Leftrightarrow q^n(x_1, x_2, \ldots, x_n).$$

Let $a \models q$. Then $tp(e/a)$ is algebraic so let $\psi(y, a)$ be a formula which isolates it. Let $\phi_n(x_1, x_2, \ldots, x_n)$ be

$$(\exists y)(\psi(y, x_1) \wedge \varphi_n(x_1, x_2, \ldots, x_n, y)).$$

Clearly ϕ_n is over $\emptyset$. We shall show:

$$(q(x_1) \wedge q(x_2) \wedge \cdots \wedge q(x_n) \wedge \phi_n(x_1, x_2, \ldots, x_n)) \Leftrightarrow q^n(x_1, x_2, \ldots, x_n).$$

Then, since ϕ_n is over $\emptyset$, the same will be true with any other $q' \in [p]^*$ in place of q (there is an automorphism moving q to q' and leaving ϕ_n fixed), proving internal isolation of p.

The $\Leftarrow$ part of the equivalence from the definition is true by construction. To prove $\Rightarrow$ assume $\models q(a_1) \wedge q(a_2) \wedge \cdots \wedge q(a_n) \wedge \phi_n(a_1, a_2, \ldots, a_n)$.

We shall show $\models q^n(a_1, a_2, \ldots, a_n)$. Let e' be such that:

$$\models \psi(e', a_1) \wedge \varphi_n(a_1, a_2, \ldots, a_n, e').$$

From $tp(a_1) = tp(a)$ and $\models \psi(e', a_1)$ we get $tp(a_1 e') = tp(ae)$. Hence there is an automorphism f of the monster such that $f(a_1 e') = ae$. Thus $f(q) = q$ and we have:

$$\models q(a) \wedge q(f(a_2)) \wedge \cdots \wedge q(f(a_n)) \wedge \varphi_n(a, f(a_2), \ldots, f(a_n), e).$$

From our assumption on q and φ_n we derive $\models q^n(a, f(a_2), \ldots, f(a_n))$ and thus $\models q^n(a_1, a_2, \ldots, a_n)$.

(b) Follows from (a).

(c) Firstly, we may absorb $acl^{eq}(\emptyset) \cap dcl(a)$ (for some $a \models p$) into the language, so that p and q become stationary. By (b) this will not affect possible internal isolation of p and q; also, it will not affect the possible smallness of T. So, assume that both p and q are stationary.

($\Rightarrow$) Suppose that p is internally isolated and we show that q is internally isolated, too. For each n let $\phi_n(x_1, x_2, \ldots, x_n)$ be a formula over $\emptyset$ such that:

$$(p(x_1) \wedge p(x_2) \wedge \cdots \wedge p(x_n) \wedge \phi_n(x_1, x_2, \ldots, x_n)) \Leftrightarrow p^n(x_1, x_2, \ldots, x_n). \qquad ((1)_n)$$

By Lemma 1.1(a) applied to p^n there are $m(n)$ and $I = b_1 b_2 \ldots b_{m(n)} \models p^{m(n)}$ such that $p^n|I \vdash p^n|B$. Let $r = stp(b_1 b_2 \ldots b_{m(n)}/B)$ and let $\varphi_n(x_1, x_2, \ldots, x_n)$ be:

$$(d_r y_1 y_2 \ldots y_{m(n)})\phi_{n+m(n)}(x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_{m(n)}).$$

We claim that φ_n, which is over $acl^{eq}(B)$, witnesses (by (a)) internal isolation of q, i.e.

$$(q(x_1) \wedge q(x_2) \wedge \cdots \wedge q(x_n) \wedge \varphi_n(x_1, x_2, \ldots, x_n)) \Leftrightarrow q^n(x_1, x_2, \ldots, x_n).$$

We show only that the left side implies the right side; the other direction is left to the reader. So assume:

$$\models q(a_1) \wedge q(a_2) \wedge \cdots \wedge q(a_n) \wedge \varphi_n(a_1, a_2, \ldots, a_n).$$

Let $I' = b'_1 b'_2 \ldots b'_{m(n)} \models r|Ba_1a_2 \ldots a_n$. By our choice of φ_n we have:

$$\models \phi_{n+m(n)}(a_1, a_2, \ldots, a_n, b'_1, b'_2, \ldots, b'_{m(n)}).$$

By $(1)_{n+m(n)}$ we have $a_1a_2 \ldots a_n b'_1 b'_2 \ldots b'_{m(n)} \models p^{n+m(n)}$. Thus $a_1a_2 \ldots a_n \models p^n|I'$. Further, from $tp(BI) = tp(BI')$ we get $p^n|I' \vdash p^n|B$ and hence $a_1a_2 \ldots a_n \models p^n|B = q^n$.

($\Leftarrow$) Suppose that q is internally isolated and, for each n, let $\varphi_n(x_1, x_2, \ldots, x_n, \bar{b})$ be a formula over $\bar{b} = B$ such that:

$$(q(x_1) \wedge q(x_2) \wedge \cdots \wedge q(x_n) \wedge \varphi_n(x_1, x_2, \ldots, x_n, \bar{b})) \Leftrightarrow q^n(x_1, x_2, \ldots, x_n).$$

Let $r = stp(\bar{b})$ and let $\phi_n(x_1, x_2, \ldots, x_n, \bar{b})$ be $(d_r\bar{y})\varphi_n(x_1, x_2, \ldots, x_n, \bar{y})$. ϕ_n is over $acl^{eq}(\emptyset)$. We show that ϕ_n and q witness internal isolation of p, by (a). Again, the $\Leftarrow$ direction follows immediately, so we prove:

$$p(x_1) \wedge p(x_2) \wedge \cdots \wedge p(x_n) \wedge \phi_n(x_1, x_2, \ldots, x_n) \Rightarrow p^n(x_1, x_2, \ldots, x_n).$$

For, suppose $a_1, a_2, \ldots, a_n \models p$ and $\models \phi_n(a_1, a_2, \ldots, a_n)$. Let $\bar{b}' \models r|a_1a_2 \ldots a_n$. Therefore, $a_1, a_2, \ldots, a_n$ realize $q' = p|\bar{b}'$ and $\models \varphi_n(a_1, a_2, \ldots, a_n, \bar{b}')$. Since q' is the conjugate of q under automorphism taking $\bar{b}$ to $\bar{b}'$ we have:

$$(q'(x_1) \wedge q'(x_2) \wedge \cdots \wedge q'(x_n) \wedge \varphi_n(x_1, x_2, \ldots, x_n, \bar{b}')) \Leftrightarrow (q')^n(x_1, x_2, \ldots, x_n).$$

Therefore $a_1a_2 \ldots a_n \models (q')^n$ and $a_1a_2 \ldots a_n \models p^n$ finishing the proof of the lemma. □

Definition 2. $p \in S(A)$ is **strictly regular** if it is regular and whenever $a, b \models p$, $a \equiv^s b\,(A)$ and $a \not\perp b(A)$ then $a = b$.

A strictly regular type is '2-internally isolated', i.e. satisfies the condition from the definition of internal isolation when $n = 2$: as a witness take formula $x_1 \neq x_2$. Actually, whenever $x_1 \neq x_2$ witnesses 2-internal isolation of a regular type, then the type must be strictly regular. In general, strict regularity (or even 'k-internal isolation' for a fixed k) does not imply internal isolation even in the case of strongly regular types. Proposition 17 from [8] provides examples of 'k-internally isolated' (k arbitrary, but fixed in advance), internally non-isolated, strongly minimal types. The most important property of strictly regular types for our purposes is the following:

Lemma 1.4. *Suppose $p, q \in S(A)$ with p strictly regular and $wt_p(q) = 1$.*

(a) *$p \overset{a}{\not\perp} q$ iff $p \in acl(q)$.*

(b) *If p is stationary, then $p \overset{a}{\not\perp} q$ iff $p \in dcl(q)$.*

(c) *If $q \in dcl(p)$ then q is strictly regular, too.*

Proof. (a) For, suppose $a \models p$, $b \models q$ and $a \not\perp b(A)$. Let $a' \models stp(a/bA)$. Then $a' \not\perp b(A)$, so $a \not\perp a'(A)$ (since $wt_p(q) = 1$) and $a = a'$ by $a \equiv^s a'(A)$ and strict regularity of p. Therefore, a is the unique realization of $stp(a/bA)$, so $a \in acl(bA)$ and $p \in acl(q)$.

(b) If p is stationary, then we pick $a'' \models tp(a/bA)$. Then $a'' \equiv^s a(A)$ and $a'' \not\perp a(A)$ imply $a = a''$, so a is the unique realization of $tp(a/bA)$ and $p \in dcl(q)$.

(c) Easy. □

In the next proposition we prove that internal isolation of a regular type is witnessed by a presence of a strictly regular type in its definable closure; alternatively, it says that internal isolation of a type is equivalent to its 2-internal isolation over all slightly larger domains.

Proposition 1.1 *(in $\mathcal{M}^{eq}$). Let A be finite and let $p \in S(A)$ be regular. Then the following conditions are all equivalent:*

(1) *p is internally isolated.*

(2) *For all finite $B \supseteq A$ and all non-forking extensions $p_1 \in S(B)$ of p there exists a strictly regular type $q \in dcl(p_1)$.*

(3) *Whenever $a_1, a_2, \ldots$ is an infinite Morley sequence in p then for all n there is a strictly regular type $q \in dcl(tp(a_{n+1}/a_1 a_2 \ldots a_n))$.*

Proof. Without loss of generality let $A = \emptyset$.

(1) $\Rightarrow$ (2) Suppose p is internally isolated, B is finite and $p_1 \in S(B)$ is a non-forking extension of p. By Lemma 1.3(c) p_1 is internally isolated so let $\phi(x_1, x_2)$ be a formula over B such that for all strong types $r \in [p_1]^*_B$:

$$(r(x_1) \wedge r(x_2) \wedge \phi(x_1, x_2)) \Leftrightarrow r^2(x_1, x_2).$$

Then, for $x_1, x_2 \models r$ we have: $x_1 \not\perp x_2$ iff $\models \neg\phi(x_1, x_2)$.

Therefore, $\neg\phi(x_1, x_2)$ is an equivalence relation on the set of realizations of each $r \in [p_1]^*_B$. By compactness, we can assume that it is a B-definable equivalence relation E on the whole monster model. We show that $q = p_1/E$ is strictly regular. Suppose $a_1/E \equiv^s a_2/E(B)$ are realizations of q, where (wlog) $a_1 \equiv^s a_2(B)$. By regularity and the above we have: $a_1/E \not\perp a_2/E(B)$ iff $a_1 \not\perp a_2(B)$ iff $\models \neg\phi(a_1, a_2)$ iff $\models E(a_1, a_2)$ iff $a_1/E = a_2/E$.

(2) $\Rightarrow$ (3) is trivial so we prove (3) $\Rightarrow$ (1).
Suppose (3) is fulfilled and let $a_1, a_2, \ldots$ be an infinite Morley sequence in p. After adding $acl^{eq}(\emptyset) \cap dcl(a_1)$ to the language we may assume that p is stationary; note that this keeps both (2) and internal isolation unharmed. By induction on n we prove that for each $k \leq n$ there exists $\phi_k(x_1, x_2, \ldots, x_k)$ over $\emptyset$ such that:

$$(p(x_1) \wedge p(x_2) \wedge \cdots \wedge p(x_k) \wedge \phi_k(x_1, x_2, \ldots, x_k)) \Leftrightarrow p^k(x_1, x_2, \ldots, x_k).$$

Suppose ϕ_k for $k \leq n$ have been already found and we find ϕ_{n+1}. Let $\bar{a} = a_1 a_2 \ldots a_{n-1}$, let $p_1 = p|\bar{a}$ and let $E(\bar{a}; x, y)$ ($=E_{\bar{a}}(x, y)$) be an equivalence relation on $p_1(\mathcal{M}^{eq})$ such that $p_1/E_{\bar{a}}$ is strictly regular. Thus for $a_n, a_{n+1} \models p_1$ we have:

$$a_n \not\perp a_{n+1}(\bar{a}) \text{ iff } a_n/E_{\bar{a}} \not\perp a_{n+1}/E_{\bar{a}}\,(\bar{a}) \text{ iff } a_n/E_{\bar{a}} = a_{n+1}/E_{\bar{a}} \text{ iff } \models E(\bar{a}; a_1, a_2);$$

the first ‘iff’ by regularity, the second by strict regularity. Therefore:

$$a_n \perp a_{n+1}(\bar{a}) \quad \text{iff} \quad \models \neg E(\bar{a}; a_1, a_2).$$

Define $\phi_{n+1}(x_1, x_2, \ldots, x_{n+1})$ to be:

$$\phi_n(x_1, x_2, \ldots, x_n) \wedge \phi_n(x_1, \ldots, x_{n-1}, x_{n+1}) \wedge \neg E(x_1, \ldots, x_{n-1}; x_n, x_{n+1}).$$

Suppose $b_1, \ldots, b_{n+1} \models p$ and $\models \phi_{n+1}(b_1, b_2, \ldots, b_{n+1})$, i.e.

$$\phi_n(b_1, b_2, \ldots, b_n) \wedge \phi_n(b_1, \ldots, b_{n-1}, b_{n+1}) \wedge \neg E(b_1, \ldots, b_{n-1}; b_n, b_{n+1}).$$

Let $\bar{b} = b_1 \ldots b_{n-1}$. Then $\models \phi_n(\bar{b}, b_n)$ and, by the induction hypothesis, $\bar{b}b_n \models p^n$; similarly $\bar{b}b_{n+1} \models p^n$. In particular $\bar{b} \models p^{n-1}$ and we have:

$$b_n \perp b_{n+1}(\bar{b}) \text{ iff } \models \neg E(\bar{b}; b_n, b_{n+1}).$$

Thus $\neg E(\bar{b}; b_n, b_{n+1})$ implies $b_n \perp b_{n+1}(\bar{b})$ and $b_1 b_2 \ldots b_{n+1} = \bar{b} b_n b_{n+1} \models p^{n+1}$. □

Corollary 1.1. *Suppose that p, q are regular, have finite domains and $p \not\perp q$. Then:*

p is internally isolated iff q is internally isolated.

Proof. By Lemma 1.3, after possibly enlarging domains and absorbing the parameters into the language, we can assume that both $p, q \in S(\emptyset)$ are stationary and $p \not\perp^{a} q$. We operate in $\mathcal{M}^{eq}$. Suppose q is internally isolated, and we show that condition (2) from Proposition 1.1 holds for p. Let B be finite and let $p_1, q_1 \in S(B)$ be non-forking extensions of p and q respectively. By Proposition 1.1 applied to q there is a strictly regular type $r \in dcl(q_1)$. Then r is stationary and $r \not\perp^{a} p_1$; by Lemma 1.4 we have $r \in dcl(p_1)$. Therefore (2) is satisfied, so p is internally isolated. □

Recall that $tp(a/A)$ is *one-based* if for all $B \supseteq A$ $tp(a/B)$ is based on aA. p is *trivial* if any pairwise independent sequence of realizations of p is independent. We need the following property shared by both one-based and trivial types (in an arbitrary stable theory):

If $p \in S(A)$ is one-based or trivial, $A \subset B$, $q \in S(B)$ is a forking extension of p and $a_1, a_2, \ldots$ is a Morley sequence in q then $a_1 \not\perp a_2(A)$.

Proposition 1.2. *A regular, internally isolated type $p \in S(\emptyset)$ which is one-based or trivial is $\not\perp$ to a U-rank 1 type.*

Proof. Without loss of generality, let $p \in S(\emptyset)$ be internally isolated and stationary. By Proposition 1.1 there is a strictly regular type $r \in dcl(p)$. We shall show that $U(r) = 1$. Let q be a forking extension of r and let $a_1, a_2, \ldots$ be a Morley sequence in q. By either one-basedness or triviality $a_1 \not\perp a_2$ Then, by strict regularity of r, we must have $a_1 = a_2$, so q must be algebraic. Any forking extension of r is algebraic, so $U(r) = 1$. $\square$

2. Strongly non-isolated types

Definition 3. Let $p \in S(A)$ be non-algebraic.

(a) p is **strongly non-isolated** if for all n and all finite B

$$\{q \in S_n(AB) \mid q \overset{a}{\perp} p\} \text{ is dense in } S_n(AB).$$

(b) p is **eventually strongly non-isolated**, or **ESN** for short, if there is a finite B and a non-forking extension $q \in S(AB)$ which is strongly non-isolated.

Note that the strong non-isolation of p is equivalent to: for all finite B and all formulas $\varphi(\bar{x})$ over AB which are consistent with T, there exists $q \in [\varphi]_{AB}$ such that $p \overset{a}{\perp} q$. If p is not strongly non-isolated then there is a finite B and consistent φ over AB such that $p \overset{a}{\not\perp} q$ for any $q \in [\varphi]_{AB}$, in which case we refer to B and φ as witnesses of the failure of strong non-isolation of p.

If p is strongly non-isolated and $\varphi(\bar{x})$ isolates a complete type $q \in S(AB)$ (where B is finite), then we must have $p \overset{a}{\perp} q$. This shows:

(1) strongly non-isolated types are $\overset{a}{\perp}$ to all isolated types over the same or slightly larger domain; in particular, they are non-isolated;

(2) an ESN type is $\perp$ to all NENI types.

If T is small then isolated types are dense in $S(A)$ for all finite A. We leave to the reader to verify:

Lemma 2.1 (T *small*). *Let $p \in S(\emptyset)$ be non-algebraic.*

(a) *p is strongly non-isolated iff $q \overset{a}{\perp}$ to all isolated types over a finite domain.*

(a)$'$ *p is ESN iff there is a finite set A and a non-forking extension $q \in S(A)$ of p such that for all finite $B \supseteq A$, $q \overset{a}{\perp}$ to all isolated types from $S(B)$.*

Let p be in addition stationary.

(b) *p is strongly non-isolated iff for all finite B if M is prime over B then $p|B \vdash p|M$.*

(b)$'$ *p is ESN iff there is a finite A such that for all finite $B \supseteq A$ if M is prime over B then $p|B \vdash p|M$.*

Lemma 2.2. *Let $p \in S(\emptyset)$. Then p is strongly non-isolated iff:*

for all finite B there is a model $M \supset B$ such that $p \overset{a}{\perp} tp(M/B)$.

Proof. We show only one direction; the other is left to the reader. Suppose p is strongly non-isolated, and B is finite. Find sequences $M = \{m_n | n \in \omega\}$ and $\{\phi_n(x) | n \in \omega\}$ such that for all $n \in \omega$:

(i) $\phi_n(x)$ is over $m_0 m_1 \ldots m_{n-1} B$ and is consistent with T,

(ii) whenever $\phi(x)$ is over M and is consistent with T then $\phi(x) = \phi_n(x)$ for some $n \in \omega$,

(iii) $\models \phi_n(m_{n+1})$,

(iv) $tp(m_n / m_0 m_1 \ldots m_{n-1} B) \overset{a}{\perp} p$.

Clearly, $M \supset B$ is a model of T and $p \overset{a}{\perp} tp(M/B)$. $\square$

In the literature there are three independently introduced notions representing properties similar to the above defined. Baisalov in [3] introduced the notion of fl-types for regular, stationary types in small, stable theories; in which context it is equivalent to our eventual strong non-isolation. His original definition used the equivalent condition from Lemma 2.1(b)′. In [2] and [3] he showed that some of the many-model constructions from the proof of Vaught's Conjecture for $\aleph_0$-stable theories from [16] still produce many countable models when $\aleph_0$-stability is weakened to superstability and when ENI types are replaced by fl-types.

Further, note that our definition makes sense even when T is uncountable. However, some of the good properties of ESN types from the countable case do not necessarily hold in the uncountable case; for example Theorem 1 fails there. If we require that the condition in the above definition of strong non-isolation holds for not only finite, but all sets B, we get Chowdhury's definition of an omissible type from [5]; there he introduces it for regular, stationary types in arbitrary (possibly uncountable) superstable theories. Chowdhury in [5] and Chowdhury and Pillay in [6] used omissible types to prove that an uncountable, complete, first-order theory T has infinitely many pairwise non-isomorphic models of size $|T|$. However, it is crucial for our purposes to demand that the condition holds only for finite sets B; otherwise the theorem is no longer true even in the weakly minimal case.

The equivalent condition from Lemma 2.1(a)′ is used in our definition of esn-types (for T small) in [17]; see also [9] and [18].

The next observation follows immediately from the definition:

Lemma 2.3. *Suppose p and q have finite domains.*
(a) *If $p \square^a q$ then p is ESN iff q is so.*
(b) *If p and q are parallel then p is ESN iff q is so.*
(c) *If p, q are regular and $p \not\perp q$ then p is ESN iff q is so.*

The following lemma, roughly speaking, says that if $\phi(x) \in p$ and p is not strongly non-isolated then a witness, say ψ, can be found so that $\models \phi \rightarrow \psi$. The lemma implies, in particular, that both strong and eventual strong non-isolation are preserved under passing from $\mathcal{M}$ to $\mathcal{M}^{eq}$ and vice versa.

Lemma 2.4. *Suppose $p \in S(\emptyset)$ and $\varphi(x) \in p$. Then p is strongly non-isolated if and only if:*

for each finite B and each formula $\phi(x)$ over B which is consistent and implies $\varphi(x)$, there exists a type $q \in [\phi]_B$ such that $p \stackrel{a}{\perp} q$.

Proof. ($\Rightarrow$) is obvious, so to prove ($\Leftarrow$) suppose that p and φ satisfy the above condition and we prove that p is strongly non-isolated. Let B be finite. As in Lemma 2.2, using the condition, one easily finds a countable model $M \supset B$ such that $p \stackrel{a}{\perp} tp(\varphi(M)/B)$. But by Proposition C.2′ from [10] we have $p \stackrel{a}{\perp} tp(M/\varphi(M)B)$, so $p \stackrel{a}{\perp} tp(M/B)$ follows and p is strongly non-isolated by Lemma 2.2. $\square$

Let A be countable and let $p \in S(A)$ be non-algebraic. Recall from [10]: p is **almost strongly regular**, or aSR for short, if there is a formula $\varphi(x) \in p$ such that for each $q \in S(AB)$ if $\varphi(x) \in q$, then either $p \perp q$ or q is a non-forking extension of p; in which case we say that p is aSR via $\varphi(x)$.

Lemma 2.5 (T *small*). *A non-isolated aSR type is strongly non-isolated. In particular, an ENI type is ESN and in the class of strongly regular types over finite domains: ESN ≡ ENI.*

Proof. Assume $p \in S(\emptyset)$ is a non-isolated aSR type. By Lemma 2.4 we may assume that it is aSR via $x = x$. For any B, if $q \in S_1(B)$ and $q \not\stackrel{a}{\perp} p$, then q is a non-forking extension of p. By the Open Mapping Theorem, since p is non-isolated, the set of all such q's is closed and nowhere dense in $S_1(B)$. Therefore, any formula over B contains a type which is $\stackrel{a}{\perp} p$; p is strongly non-isolated. $\square$

Example 3. A regular eni type, even in an $\aleph_0$-stable theory, does not have to be necessarily ESN; the reason for this is a possible non-orthogonality to a NENI type. Here is an example:

Let $A = (\omega + 1) \times (\omega + 1)$. For $(i, j), (k, l) \in A$ define:

$E((i, j), (k, l))$ iff $i = k$; $R_n((i, j))$ iff $j = n$.

Let $T = Th(A, E, R_n)_{n\in\omega}$. We leave to the reader to verify that T is $\aleph_0$-stable, $tp(1, \omega)$ is regular, non-isolated (and thus eni) but is not ESN (because $(1, \omega)$ forks with $(1, 1)$ and $tp(1, 1)$ is NENI).

Now, we give a topological characterization of strong non-isolation. Let $p \in S(\emptyset)$, let A be finite and let $q = tp(\bar{a}/A)$ be a non-forking extension of p. Suppose that $\phi(x, \bar{a})$ forks over A. Then the set

$$Tr_A(\phi(x, \bar{a})) = \{r \in S(A) \mid r(x) \text{ is consistent with } \phi(x, \bar{a})\}$$

is a closed subset of $S(A)$ and hence $\{r \in S(A) \mid r \not\perp^{a} q\}$ is a union of countably many closed sets. We have the following two possibilities:

(I) $\{r \in S_n(A) \mid r \not\perp^{a} q\}$ is meager for all n; or

(II) for some $\phi(x, y)$, which is over A, $\phi(x, \bar{a})$ forks over A and $Tr_A(\phi(x, \bar{a}))$ has non-empty interior in $S(A)$.

By the Baire Category Theorem the first possibility happens to hold for all finite A exactly when p is strongly non-isolated. (II) is a useful consequence of the failure of strong non-isolation. We have just proved:

Lemma 2.6. *Suppose $p \in S(\emptyset)$ is stationary. The following conditions are all equivalent:*

(a) *p is strongly non-isolated.*

(b) *For all finite B and $a \models p|B$: $Tr_B(\phi(x, a))$ is nowhere dense in $S(B)$ for all $\phi(x, a)$ forking over B (where $\phi(x, y)$ is over B).*

(c) *There does not exist a finite set B and formulas $\phi(x)$ and $\varphi(x, y)$ over B such that $a \models p|B$, $\varphi(x, a)$ forks over B and $Tr_B(\varphi(x, a)) \supseteq [\phi]_B \neq \emptyset$.*

(d) *For all finite B $\{q \in S_n(B) \mid q \perp^{a} p\}$ is meager in $S_n(B)$ for all n.*

Note the similarity of (d) with the condition from the definition of meager forking; see [12].

Proposition 2.1. *A finite product of stationary, strongly non-isolated types over the same domain is strongly non-isolated. A finite product of ESN strong types over arbitrary finite domains is ESN.*

Proof. Let $p, q \in S(\emptyset)$ be stationary with q strongly non-isolated. Assuming that $p \otimes q$ is not strongly non-isolated we shall show that p is not strongly non-isolated.

Fix $a \models p$ and $b \models q$ with $a \downarrow b$. Since $tp(ab)$ is not strongly non-isolated, by Lemma 2.6(c) there are formulas $\varphi(x, y, \bar{z})$ and $\phi(\bar{z})$ (wlog over $\emptyset$) such that $\varphi(a, b, \bar{z})$ forks over $\emptyset$ and:

$$Tr(\varphi(a, b, \bar{z})) = \{r \in S(\emptyset) \mid r(\bar{z}) \wedge \varphi(a, b, \bar{z}) \text{ is consistent}\} \supseteq [\phi] \neq \emptyset.$$

Let $p_1 = p|b$. We show that $r \in [\phi(\bar{z}) \wedge (d_{p_1}x)\varphi(x, b, \bar{z})]_b$ implies that r forks over $\emptyset$: if $\bar{c}' \models r$ and $a' \models p_1$ are such that $\bar{c}' \downarrow a'(b)$ and $\models \varphi(a', b, \bar{c}')$, then $\bar{c}' \not\downarrow a'b$ (since $\varphi(a', b, \bar{z})$ forks over $\emptyset$) which combined with $\bar{c}' \downarrow a'(b)$ implies $\bar{c}' \not\downarrow b'$. Therefore if $\phi(\bar{z}) \wedge (d_{p_1}x)\varphi(x, b, \bar{z})$ is consistent then it forks over $\emptyset$.

But $q = tp(b)$ is strongly non-isolated, so by Lemma 2.6(b):

$$F = Tr(\phi(\bar{z}) \wedge (d_{p_1}x)\varphi(x, b, \bar{z})) \text{ is nowhere dense in } S(\emptyset).$$

Thus $tp(\bar{c}) \in [\phi] \setminus F$ and $\models \phi(a, b, \bar{c})$ imply $\bar{c} \not\downarrow a(b)$, since $\phi(a, b, \bar{z})$ forks over $\emptyset$. Also, F is closed and nowhere dense, so let $\psi(\bar{z})$ be over $\emptyset$ such that $[\phi] \setminus F \supseteq [\psi] \neq \emptyset$. Then $\psi(\bar{z}) \wedge \varphi(a, b, \bar{z})$ forks over b and $Tr_b(\psi(\bar{z}) \wedge \varphi(a, b, \bar{z})) = [\psi]_b \neq \emptyset$. By Lemma 2.6(c) this means that p is not strongly non-isolated.

We have just proved the case $n = 2$. The general case follows by induction. □

Lemma 2.7. *Suppose $\{p_m \mid m \in \omega\} \subseteq S(\emptyset)$ is a family of stationary, strongly non-isolated types. Then $\{q \in S_n(\emptyset) \mid q \perp^{a} p_m^{\omega}$ for all $m\}$ is dense in $S_n(\emptyset)$ for all n.*

Proof. By Proposition 2.1 each p_m^n is strongly non-isolated so, by Lemma 2.6(d), each $E_{n,m} = \{q \in S_n(\emptyset) \mid q \not\perp^{a} p_m^n\}$ is meager. Then $E = \bigcup\{E_{n,m} \mid m, n \in \omega\}$ is meager, too. By the Baire Category Theorem the set $S_n(\emptyset) \setminus E$ is dense in $S_n(\emptyset)$ and any $q \in S_n(\emptyset) \setminus E$ satisfies $q \perp p_m^{\omega}$ for all $m \in \omega$. □

The following is an immediate consequence of the previous lemma:

Proposition 2.2. *Suppose that A is countable and $\{p_n | n \in \omega\} \subseteq S(A)$ is a family of stationary, strongly non-isolated types. Then there is a model $M \supseteq A$ such that $tp(M/A) \perp^{a} p_n^{\omega}$ for all $n \in \omega$.*

3. The superstable case

In this section we prove the superstable case of the main theorem. So throughout we assume that T is countable and superstable. We have to show that for a regular type $p \in S(\emptyset)$ the following conditions are all equivalent:

(A) p is ESN.

(B) For each finite B and each stationarization p' of p there exists a model $M \supseteq B$ such that $dim(q, M)$ is finite for all regular, stationary types $q \not\perp p'$ whose domain is a finite subset of M.

(C) For all NENI types r whose domain is a finite subset of $\mathcal{M}$ we have $p \perp r$.

A few words about the proof. (A)⇒(B) follows rather easily from the results from Section 2 and (B)⇒(C) is straightforward. The main body of the proof is (C)⇒(A) (actually, ¬(C)⇒ ¬(A)). Assuming that p is not ESN we have to find a NENI type $q \not\perp p$. Working within a $wt_p = 1$ formula φ we prove in Lemma 3.1 that p is internally isolated. Then, assuming that p is stationary, forking of a realization of p with an element of $\varphi(\mathcal{M})$ can be witnessed by a presence of a strictly regular type in $dcl(p)$. Using the Baire Category Theorem we find a uniform way of finding such a strictly regular type on an open subset of $[\varphi]$. This ensures that the strictly regular type is internally isolated, isolated and stationary; it must be NENI.

Proof. (A)⇒(B) Let $p \in S(\emptyset)$ be ESN and let B be finite. Let $C \supseteq B$ be finite and let $p' \in S(C)$ be a stationary, strongly non-isolated, non-forking extension of p. By Proposition 2.2 there is a model $M \supseteq C$ such that $tp(M/C) \overset{a}{\perp} (p')^{\omega}$. We show that M satisfies condition (B).

Let $dom(q) = D \subset M$ be finite, where q is regular, stationary and $q \not\perp p'$. Choose $m \in \omega$ such that $(q|CD)^m \overset{a}{\not\perp} (p'|CD)^{\omega}$. Combining with $tp(M/C) \overset{a}{\perp} (p')^{\omega}$ we get $dim(q|CD, M) < m$. Then $dim(q, M) < \aleph_0$ follows from the superstability. □

Proof. (B)⇒(C) Suppose (C) is not true and let $p \not\perp q$ where $q \in S(B)$ is NENI and B is finite. Then for every model $M \supseteq B$ we have $dim(q, M) \geq \aleph_0$ contradicting (B). □

Lemma 3.1. *A regular type (over $\emptyset$) which is not ESN must be internally isolated.*

Proof. Without loss of generality we operate in $\mathcal{M}^{eq}$. Suppose $p \in S(\emptyset)$ is a regular type which is not ESN. We shall show that p is internally isolated. Since both internal and eventual strong non-isolation are properties of $\not\perp$-classes of regular types over finite domains, after naming a few parameters and replacing p by a type from its $\not\perp$-class, we may assume that $R^{\infty}(p)$ is minimal possible for the types in the class; also, we may assume that p is stationary. Let $\varphi(x) \in p$ be such that $R^{\infty}(\varphi) = R^{\infty}(p)$. Note that whenever q' is a forking extension of some $q \in [\varphi]$ then $R^{\infty}(\varphi) > R^{\infty}(q')$ so $wt_p(q') = 0$ by the minimality assumption. This shows that any $q \in [\varphi]^*$ either has p-weight 0 or is regular $\not\perp p$.

We shall prove that condition (2) from Proposition 1.1 is satisfied.

Claim 1. For all finite B there is a finite set $C \supset B$ and formulas $\psi(y)$ and $\phi(x, y)$ over C such that:
(a) $\emptyset \neq [\psi]_C \subseteq [\varphi]_C$ and $\models \phi(y, x) \to (\varphi(x) \wedge \varphi(y))$;
(b) $\psi(y) \wedge \phi(y, a)$ is consistent for any $a \models p|C$;
(c) $\models \psi(b)$ implies $\phi(b, x)$ forks over C (and thus $wt_p(\phi(b, x)) = 0$);
(d) each $q \in [\psi]^*_C$ is regular and $\not\perp p$.

Since p is not ESN there is a finite subset $C \supseteq B$ and a formula $\psi'(y)$ over C which is consistent with T such that $q \overset{a}{\not\perp} p$ whenever $q \in [\psi']_C$. By Lemma 2.4 we may assume that $\models \psi' \to \varphi$.

For $q \in [\psi']_C$ let $b \models q$ and $a \models p|C$ satisfy $a \not\perp b(C)$. Witness the dependence by a formula $\theta_q(b, x)$ which forks over C, implies $\varphi(x) \wedge \varphi(y)$ and is such that $\models \theta_q(b, a)$ and $\theta_q(y; x) \notin bnd_C(q(y))$.

For each formula $\theta(y, x)$ over C implying $\varphi(x) \wedge \varphi(y)$ define:

$$E_{\theta} = \{q \in [\psi']_C \mid \theta(y; x) \notin bnd_C(q(y)) \text{ and } \theta(y, x) \text{ consistent with } q(y) \cup p(x)|C\}.$$

Both conditions '$\notin bnd$' and 'is consistent with' define closed subsets, so each E_{θ} is a closed subset of $[\psi']_C$. By the above considerations:

$$\bigcup\{E_{\theta} | \theta(y, x) \text{ is over } C \text{ and implies } \varphi(x) \wedge \varphi(y)\} = [\psi']_C.$$

The Baire Category Theorem applies and we get formulas $\phi(y, x)$ (which implies $\varphi(x) \wedge \varphi(y)$) and $\psi(y)$ over C with $\emptyset \neq [\psi]_C \subseteq E_\phi$. Condition (a) is satisfied since $[\psi]_C \subseteq [\psi']_C \subseteq [\varphi]_C$; (b) is satisfied by the consistency condition in E_ϕ; (c) is satisfied since $\phi(y; x) \notin bnd_C(q(y))$ for any $q \in [\psi]_C$ and $wt_p(\phi(b, x)) = 0$ follows from $R^\infty(\phi(b, x)) < R^\infty(p)$. To verify (d) note that whenever $tp(b) \in [\psi]_C \subseteq E_\phi$ then b can fork with a realization of $p|C$ by the consistency condition from the definition of E_ϕ, so $stp(b/C) \not\perp p$; by our choice of φ, we must have that $stp(b/C)$ is regular, completing the proof of Claim 1.

Continuing the proof of the lemma, let D be finite and let $a_1 \models p|D$. By Claim 1 (applied to p and $B = a_1 D$), there is $\bar{c} \supseteq a_1 D$ and formulas ϕ and ψ over $\bar{c}$ satisfying:

(i) $\emptyset \neq [\psi]_{\bar{c}} \subseteq [\varphi]_{\bar{c}}$ and $\models \phi(y, x) \to (\varphi(x) \wedge \varphi(y))$;
(ii) $\psi(y, \bar{c}) \wedge \phi(y, a, \bar{c})$ is consistent for $a \models p|\bar{c}$;
(iii) $\models \psi(b, \bar{c})$ implies $\phi(b, x, \bar{c})$ forks over $\bar{c}$ (and thus $wt_p(\phi(b, x, \bar{c})) = 0$);
(iv) each $q \in [\psi]^*_{\bar{c}}$ is regular and $\not\perp p$;

Let $r = stp(\bar{c}/a_1 D)$. Consider the formula: $(d_r \bar{z})(\exists y)(\psi(y, \bar{z}) \wedge \phi(y, x, \bar{z}))$. It is over $acl(a_1 D)$ so denote it by $\sigma(x, a_1, e)$ where $\sigma(x, x_1, t)$ is over D and $e \in acl(a_1 D)$. We prove that σ witnesses 2-internal isolation of $p|D$:

Claim 2. For $a \models p|D$: $a \perp\!\!\!\perp a_1(D)$ iff $\models \sigma(a, a_1, e)$.

If $a \models p|a_1 D$ then by (ii) $\models \sigma(a, a_1, e)$. Conversely, suppose $a \models p|D$ and $\models \sigma(a, a_1, e)$. Let $\bar{c}' \models r$ and $\bar{c}' \perp\!\!\!\perp a(a_1 D)$. Pick b' such that $\models \psi(b', \bar{c}') \wedge \phi(b', a, \bar{c}')$. Since $tp(\bar{c}) = tp(\bar{c}')$ (iii) holds with $\bar{c}'$ in place of $\bar{c}$, so we have that $\phi(b', x, \bar{c}')$ forks over $\bar{c}'$ and hence $a \not\perp\!\!\!\perp b'(\bar{c}')$. But by (iv) (with $\bar{c}'$ in place of $\bar{c}$) we have $stp(b'/\bar{c}') \square p$ which implies $wt_p(a/\bar{c}') = 1$. Thus $a \models p|\bar{c}'$ and $a \models p|a_1 D$, completing the proof of Claim 2.

Since p is stationary, the left-hand side of the equivalence in Claim 2 (considered as a formula in variable a) is invariant under $a_1 D$-automorphisms, so $\sigma(x, a_1, e)$ can be replaced by a formula over $a_1 D$ keeping Claim 2 true. Thus $p|D$ is 2-internally isolated. As in the proof of Proposition 1.1 we derive that there is a strictly regular type in $dcl(p|D)$. Thus the condition (2) from Proposition 1.1 is satisfied and p is internally isolated. $\square$

Proof. (C)⇒(A) Let $p \in S(\emptyset)$ be a stationary, regular type which is not ESN. We shall find a NENI type in $\mathcal{M}^{eq}$ which is $\not\perp p$. Throughout, we operate in $\mathcal{M}^{eq}$. Let $\varphi(x)$ be a formula of minimal R^∞-rank contained in some type which is $\not\perp p$. Thus, whenever $q \in [\varphi]$ then q is either regular $\not\perp p$, or $wt_p(q) = 0$. Further, after naming some parameters and replacing p by a type from its $\not\perp$-class, we may assume that $\varphi(x) \in p$.

p is not strongly non-isolated so we can find a consistent formula $\psi(x)$ (wlog over $\emptyset$) as a witness: $p \not\perp^a q$ for all $q \in [\psi]$. Moreover, by Lemma 2.4 we may assume $[\psi] \subseteq [\varphi]$. By Lemma 3.1 p is internally isolated, so by Proposition 1.1 there is a strictly regular type $r \in dcl(p)$. We *claim* that r is a NENI type. Since p is stationary and internally isolated so is r and, by Lemma 1.2, it remains to show that r is isolated.

For each $q \in [\psi]$ we have $q \not\perp^a p$, so $r \not\perp^a q$ and, since r is strictly regular and stationary, by Lemma 1.4, we must have $r \in dcl(q)$. Thus, for each $q \in [\psi]$ there is a 'formula' $x = f(y)$ witnessing $r \in dcl(q)$.

Let $\mathcal{F}$ be the set of all formulas of the form $x = f(y)$ where f is a $\emptyset$-definable partial function. Fix $a \models r$ and for $(x = f(y)) \in \mathcal{F}$ let $C_f = [\psi] \cap Tr(a = f(y))$. Clearly, each C_f is closed and, by the above consideration, we have:

$$\bigcup \{C_f \mid x = f(y) \in \mathcal{F}\} = [\psi].$$

But $\mathcal{F}$ is countable, so by the Baire Category Theorem at least one of the C_f's has non-empty interior. Fix $x = f(y) \in \mathcal{F}$ and $\tau(y)$ such that $\emptyset \neq [\tau] \subseteq C_f$. We show that $(\exists y)(\tau(y) \wedge x = f(y))$ isolates r.

Suppose $\models (\exists y)(\tau(y) \wedge a' = f(y))$ and we show that $a' \models r$. For some b' we have $\models \tau(b') \wedge a' = f(b')$. Since $[\tau] \subseteq C_f$ there is $b \equiv^s b'$ such that $a = f(b)$. We conclude that $a = f(b) \equiv^s f(b') = a'$ and hence a' is a realization of r. Therefore $(\exists y)(\tau(y) \wedge x = f(y))$ isolates r, completing the proof of the theorem. $\square$

(A)⇔(C) in the main theorem is proved only for regular types p; however it is not hard to see that the equivalence holds for all p.

Corollary 3.1. *A type $p \in S(\emptyset)$ is ESN if and only if $p \perp q$ for all NENI types q whose domain is a finite subset of $\mathcal{M}^{eq}$.*

Proof. Let $p \square^a p_1 \otimes p_2 \otimes \cdots \otimes p_n$ where $p_1, p_2, \ldots, p_n$ are regular, stationary types (wlog over $\emptyset$). Then p is ESN iff $p_1 \otimes p_2 \otimes \cdots \otimes p_n$ is ESN iff (by Proposition 2.1) all p_i's are ESN iff all p_i's are $\perp$ all NENI types over finite domains iff $p \perp$ all NENI types. $\square$

Corollary 3.2. (a) *If $tp(a)$ is non-isolated then there is a finite set C such that $tp(a/C)$ is strongly non-isolated.*
(b) *If T is not $\aleph_0$-categorical then there exists a strongly non-isolated type over a finite domain.*

Proof. (a) We shall find the desired C in $\mathcal{M}^{eq}$. Let C be finite such that $tp(a/C)$ is non-isolated and $U(a/C)$ is minimal possible. If $tp(a/C)$ is not ESN then, after possibly slightly enlarging C, we find b with $tp(b/C)$ NENI and $a \not\perp b(C)$. Then $tp(a/bC)$ is non-isolated and $U(a/bC) < U(a/C)$, contradicting the minimality assumption. Thus $tp(a/C)$ is ESN; to make it strongly non-isolated enlarge C if necessary and replace it by a subset of $\mathcal{M}$.
(b) Follows from (a). $\square$

As an immediate consequence of Proposition 1.2 and the theorem we have:

Proposition 3.1. *A regular type of limit-ordinal U-rank, which is one-based or trivial, must be ESN. (Alternatively: any NENI type which is one-based or trivial must be $\not\perp$ to a strongly minimal type.)*

4. The small stable case

This section is devoted to the proof of Theorem 1 in the small, stable case. The plan of the proof is almost the same as in the superstable case. Again, (C)⇒(A) is the difficult part; assuming that p is not ESN we find a NENI type $\not\perp p$. First we find a $wt_p = 1$ formula and then working within it we show that p is internally isolated; further, we witness a possible forking with a realization of an isolated type of $wt_p = 1$ by a formula algebraizing a strictly regular type from $acl(p)$. The strictly regular type is isolated, internally isolated and stationary, and hence is NENI.

Throughout the section assume that T is small and stable.

Lemma 4.1. *Suppose M is prime, $C \subset M$ is finite, $q \in S(C)$ is a stationary ESN type of finite weight, and $\{a_0, a_1, \ldots\}$ is an infinite Morley sequence in q. Then for some $n \in \omega$ $tp(a_n/M)$ is a non-forking extension of q. In particular, $dim(p, M) < \aleph_0$ for all regular ESN types $p \not\perp q$ whose domain is a finite subset of M.*

Proof. Let $A \supseteq C$ be finite such that $q|A$ is strongly non-isolated and let M_1 be prime over A. Since q is stationary A and M_1 can be found so that $M \subseteq M_1$. Apply Lemma 1.1(b) to q and A (with C absorbed into the language). Thus $a_n \downarrow A(C)$ for some $n \in \omega$. Now $tp(a_n/A) = q|A$ is strongly non-isolated so, since M_1 is atomic over A, we have $a_n \downarrow M_1(A)$. Therefore $a_n \downarrow M_1(C)$ and $a_n \downarrow M(C)$. $\square$

Proof. **(A)⇒(B)** Let $p \in S(\emptyset)$ be a regular ESN type, let B be finite and let $M \supset B$ be prime over B. By Lemma 4.1 $dim(q, M) < \aleph_0$ for each regular, stationary ESN type over a finite subset of M. $\square$

(B)⇒(C) is trivial, so the rest of the section is devoted to the proof of **(C)⇒(A)**.

Without loss of generality let $\mathcal{M} = \mathcal{M}^{eq}$. Suppose $p \in S(\emptyset)$ is regular and stationary but is not ESN. We shall find a NENI type over a finite domain which is $\not\perp p$.

Claim 1. There exists a p-simple formula of p-weight 1.

Proof. First of all note that there is an isolated type of positive p-weight: since p is not ESN there is an isolated type q' such that $q' \not\perp p$; then there is a p-internal type $q \in acl(q')$ witnessing $q' \not\perp p$. Clearly q is isolated and has positive p-weight.

Let $n \geq 1$ be the minimal possible positive p-weight of a p-simple formula and let $\varphi(x)$ be one such formula. After adding a few parameters to the language, we can find $a \models p$ and b such that $\models \varphi(b)$, $wt_p(b) = n$ and $a \not\downarrow b$. Clearly $wt_p(b/a) = n - 1$. If $\psi(x, y)$ witnesses $a \not\downarrow b$ then $wt_p(\psi(x, a) \wedge \varphi(x)) = n - 1$. By minimality of n we have $n - 1 = 0$, proving the claim. $\square$

By Lemma 2.4 and Claim 1 we may from now on assume $wt_p(x = x) = 1$.

Claim 2. p is internally isolated.

Proof. We shall prove that condition (2) from Proposition 1.1 is satisfied. Let $\bar{b}$ be arbitrary and let $p_1 \in S(\bar{b})$ be the non-forking extension of p. We shall show that there is a strictly regular type $q \in dcl(p_1)$.

Let $a_1 \models p_1$. Since $p_1|a_1\bar{b}$ is not strongly non-isolated, there are $a_2, \bar{c}, d$ with $\bar{c} \supseteq a_1\bar{b}$, $tp(d/\bar{c})$ isolated, $a_2 \models p|\bar{c}$ and $a_2 \not\perp d(\bar{c})$.

Let $E = cl_p(\bar{c})$. Then $a_2 \perp E$, $tp(d/E)$ is $\square^a$ to a power of p and $a_2 \not\perp d(E)$ (for details see [7]). Find $e \in E$ and ψ (over $\emptyset$) witnessing the dependence:

$$\models \psi(d, a_2, \bar{c}, e) \text{ and } \psi(t; x_2, \bar{z}, u) \notin bnd(tp_t(d/E)).$$

Choose $\phi(t, \bar{c})$ isolating $tp(d/\bar{c})$ and let $r = stp(\bar{c}e/a_1 b)$. Consider the formula:

$$(d_r \bar{z} u)(\exists t)(\phi(t, \bar{z}) \wedge \psi(t, x_2, \bar{z}, u)).$$

It is over $acl(a_1\bar{b})$, so denote it as $\theta(x_2, a_1, \bar{b})$. We show that it witnesses 2-internal isolation of p_1, i.e. that for $a_2' \models p_1$ we have:

$$\models \theta(a_2', a_1, \bar{b}) \text{ if and only if } a_2' \perp a_1(\bar{b}).$$

$\models \theta(a_2, a_1, \bar{b})$ proves the 'only if' part. For the 'if' part, suppose $a_2' \models p_1$ and $\models \theta(a_2', a_1, \bar{b})$. Pick $\bar{c}'e' \models r$ with $\bar{c}'e' \perp a_2'(a_1\bar{b})$, and d' such that:

$$\models \phi(d', \bar{c}') \wedge \psi(d', a_2', \bar{c}', e').$$

Let $E' = cl_p(\bar{c}')$. Since $\bar{c}' \equiv^s \bar{c}(\bar{b}a_1)$ we have $E' \equiv^s E(\bar{b}a_1)$. Then $d'E' \equiv dE(a_1\bar{b})$ (since $tp(d'/\bar{c}')$ is isolated by $\phi(t, \bar{c}')$) and thus $\psi(t; x_2, \bar{z}, u) \notin bnd(tp_t(d'/E'))$ which together with $\models \psi(d', a_2', \bar{c}', e')$ implies $d' \not\perp a_2'(E')$. But $tp(d'/E')$ is $\square^a$ to a power of p, so we conclude that $tp(a_2'/E')$ has positive p-weight. The last implies that $tp(a_2'/E')$ is a non-forking extension of p_1, completing the proof of the above equivalence.

As in the proof of Lemma 3.1 we conclude, first that $\theta(x, a_1, \bar{b})$ can be chosen over $a_1\bar{b}$, and then that there is a strictly regular type $q \in dcl(p_1)$. By Proposition 1.1 p is internally isolated. $\square$

p is not ESN so, by Lemma 2.4, there is a finite set A and an isolated type $q \in S_1(A)$ such that $p \not\perp^a q$. Then $wt_p(x = x) = 1$ implies $wt_p(q) = 1$. By Lemma 1.3(c) $p|A$ is internally isolated, so there exists a strictly regular type $r \in dcl(p|A)$. Clearly, r is internally isolated, stationary and $r \not\perp^a q$. By Lemma 1.4(b) this implies $r \in dcl(q)$, so r is isolated and by Lemma 1.2 r is NENI. The proof of the theorem is now complete.

Combining Proposition 1.2 and the theorem we obtain:

Corollary 4.1. *Let A be finite and let $p \in S(A)$ be one-based or trivial, and regular. If p is $\perp$ to all U-rank 1 types, then p is ESN. (Alternatively: if p is NENI then $p \not\perp$ to a strongly minimal type.)*

5. Lascar's rank and the number of countable models

In this section we prove a special case of:

Conjecture. If T is superstable and $U(T) \geq \omega^\omega$ then $I(T, \aleph_0) = 2^{\aleph_0}$.

Although Vaught's Conjecture for $\aleph_0$-stable theories has been proved (in [16]), we were not able to prove the $\aleph_0$-stable case of the conjecture. Moreover, it seems that assuming $\aleph_0$-stability of the theory does not simplify the situation at all; the problem seems to be of geometric nature.

Our plan of the proof is as follows: assuming T superstable and $U(T) \geq \omega^\omega$, one finds an infinite set of integers n and for each of them an ESN type of U-rank ω^n; since these types are orthogonal in a strong sense (i.e. any two conjugates of different types are orthogonal) and are ESN, varying their dimensions one constructs continuum pairwise non-isomorphic countable models. The first part we were able to do only in some special cases; the second is done in Proposition 5.1 below (part (2) of which was proved by Baisalov in [2]).

Recall that B is almost atomic over A if for each $\bar{b} \subseteq B$ and finite $A_0 \subseteq A$ there is finite A' such that $A_0 \subseteq A' \subseteq A$ and $tp(\bar{b}/A')$ is isolated. If T is small then almost atomic models over arbitrary sets exist.

Throughout the rest of the section assume that T is small and stable.

Lemma 5.1. *Let $p \in S(\emptyset)$ be strongly non-isolated and let B be almost atomic over A. Then $p \stackrel{a}{\perp} tp(B/A)$.*

Proof. Otherwise there would be $c \models p$, $c \downarrow A$ and $\bar{b} \subseteq B$ such that $c \not\downarrow \bar{b}(A)$. Then $c \not\downarrow \bar{b}A_0$ for some finite $A_0 \subseteq A$. Since $\bar{b}$ is almost atomic over A there is finite A' such that $A_0 \subset A' \subset A$ and $tp(\bar{b}/A')$ is isolated. Clearly $c \not\downarrow \bar{b}A'$ and combining with $c \downarrow A$ we get $c \not\downarrow \bar{b}(A')$ contradicting the strong non-isolation of $tp(c/A_1)$. □

Proposition 5.1. *Suppose that $\{p_n | n \in \omega\}$ is a family of ESN types of finite weight such that $dom(p_n)$ is finite and $p_n \perp$ to every conjugate of p_m for all $m \neq n$. Then $I(T, \aleph_0) = 2^{\aleph_0}$ provided that at least one of the following conditions holds:*

(1) *$p_n \perp \emptyset$ for all $n \in \omega$;*

(2) *T is superstable.*

Proof. (1) Without loss of generality T is small, p_n is stationary and strongly non-isolated and $B_n = dom(p_n)$ is finite for all $n \in \omega$. Further, assume that $\{B_n | n \in \omega\}$ is independent over $\emptyset$; to justify this assumption, note that if we replace p_n by a conjugate of itself then the conditions of the lemma remain valid. Let $B = \bigcup\{B_n | n \in \omega\}$ and let $M \supseteq B$ be a countable model, almost atomic over B.

Let $X \subseteq \omega$ be arbitrary. We shall construct a countable model M_X such that $m \in X$ if and only if:

$m \in \omega$ and for all $C \subseteq M_X$ and all $p \in S(C)$ if $tp(C) = tp(B_m)$ and p is a conjugate of p_m,
then in M_X there exists an infinite Morley sequence in p.

Inductively define a sequence of countable models $\{M_X^n | n \in \omega\}$. Let $M_X^0 = M$. Suppose that M_X^n has already been constructed. Let $\mathcal{F}_n$ be the set of conjugates of p_m's for all $m \in X$ whose domain is a finite subset of M_X^n. Further, note that $\mathcal{C}_n = \{dom(p) | p \in \mathcal{F}_n\}$ is countable since M_X^n is countable. Since T is small for all $C \in \mathcal{C}_n$ there are at most countably many $p \in \mathcal{F}_n$ such that $dom(p) = C$. Therefore $\mathcal{F}_n$ is countable. For $p \in \mathcal{F}_n$ choose a countable Morley sequence in $p|M_X^n$ and call it I_p. Moreover, assume that our choice is such that $I_n = \bigcup\{I_p | p \in \mathcal{F}_n\}$ is independent over M_X^n. Let M_X^{n+1} be a countable model almost atomic over $I_n M_X^n$, and let $M_X = \bigcup\{M_X^n | n \in \omega\}$.

To prove the condition above notice that, by construction, it suffices to show that p_k is not realized in M_X for all $k \in \omega \setminus X$. Fix one such k. Since $p_k \perp \emptyset$ and B is independent over $\emptyset$ we have $p_k \vdash p_k | B$. Since M is almost atomic over B and p_k is strongly non-isolated, by Lemma 5.1 we have $p_k \stackrel{a}{\perp} tp(M/B)$ and hence $p_k \vdash p_k | M$. Since I_0 is an independent set of realizations of types which are $\perp p_k$, we derive $p_k \perp tp(I_0/M)$. Further, M_X^1 is almost atomic over $I_0 M$, so $p_k \stackrel{a}{\perp} tp(M_X^1/I_0 M)$ by Lemma 5.1 and hence $p_k | M \vdash p_k | M_X^1$. Continuing in this way we get $p_k | M_X^1 \vdash p_k | M_X^2 \ldots$. Altogether $p_k \vdash p_k | M_X$ and p_k is not realized in M_X.

For $X, Y \subseteq \omega$ and $X \neq Y$ we have $M_X \not\cong M_Y$ and $I(T, \aleph_0) = 2^{\aleph_0}$ follows.

(2) Without loss of generality T is small and superstable. After replacing p_n by one of its regular components we may assume that each p_n is regular. By (1) we may also assume that $p_n \not\perp \emptyset$ holds for all $n \in \omega$. Let M be a prime model. Now we find a sequence of regular types $\{r_n \mid n \in \omega\}$ such that $dom(r_n) = B_n$ is a finite subset of M, $r_n \not\perp$ to a conjugate of p_n, and each r_n either is stationary or else is non-isolated and almost strongly regular.

By Theorem D.17 from [10], for each $n \in \omega$ there exists a regular type $q_n' \in S(M)$ such that $q_n' \not\perp p_n$ and q_n' is a non-forking extension of $q_n = q_n' | A_n$, where $A_n \subseteq M$ is finite and q_n is almost strongly regular. If q_n is isolated then it is realized in M by b_n say, so let $B_n = b_n A_n$ and let $r_n \in S(B_n)$ be the non-forking extension of $stp(b_n/A_n)$; note that r_n is stationary and $\not\perp$ to a conjugate of p_n. If q_n is non-isolated let $r_n = q_n$. Clearly, each r_n is ESN and for $n \neq m$ $r_n \perp$ each conjugate of r_m.

Fix $X \subseteq \omega$ and construct M_X as in (1). Inductively define a sequence of countable models $\{M_X^n | n \in \omega\}$. Let $M_X^0 = M$ and suppose that M_X^n has already been constructed. Let $\mathcal{F}_n$ be the set of all conjugates of r_m's (where $m \in X$) whose domain is a finite subset of M_X^n. Then $\mathcal{F}_n$ is countable and for $p \in \mathcal{F}_n$ choose a countable Morley sequence I_p in a non-forking extension of p to M_X^n. Moreover, assume that our choice is such that $I_n = \bigcup\{I_p | p \in \mathcal{F}_n\}$ is independent over M_X^n. I_n is countable, so let M_X^{n+1} be a countable model almost atomic over $I_n M_X^n$ and let $M_X = \bigcup\{M_X^n | n \in \omega\}$. We shall prove that $m \in X$ if and only if:

$m \in \omega$ and for all $C \subseteq M_X$ and $p \in S(C)$ if $tp(C) = tp(B_m)$ and p is a conjugate of r_m,
then in M_X there exists an infinite Morley sequence in p.

By construction, it is enough to prove that for $m \in \omega \setminus X$ there does not exist in M_X an infinite Morley sequence in r_m. Suppose that $\{a_1, a_2, \ldots\}$ is an infinite Morley sequence in r_m. We show that $a_k \notin M_X$ for some $k \in \omega$. First we show that $tp(a_k/M)$ is a non-forking extension of r_m for some $k \in \omega$. We have the following two subcases:

Subcase 1. r_m is almost strongly regular and non-isolated.

r_m is strongly non-isolated by Lemma 2.5. Then $r_m \overset{a}{\perp} tp(M/B_m)$ since M is atomic over B_m and $tp(a_k/M)$ is a non-forking extension of r_m for all k.

Subcase 2. r_m is stationary.

Since M is atomic over B_m Lemma 4.1 applies and $tp(a_k/M)$ is the non-forking extension of r_m for some k.

As in (1) we have $tp(a_k/M) \vdash tp(a_k/M_X)$ which combined with $a_k \mathop{\smile\!\!\!\!\!\!\perp} M(B_m)$ implies $a_k \notin M_X$.

Finally, for $X, Y \subseteq \omega$ and $X \neq Y$ we have $M_X \ncong M_Y$ so $I(T, \aleph_0) = 2^{\aleph_0}$. □

The orthogonality condition in the previous proposition cannot be weakened in general. The example of abnormal type from [1] XVIII.4 shows that the assumption is necessary in case (1) even if T is $\aleph_0$-stable.

In the next theorem $U(T)$ denotes $sup\{U(p) \mid p \in S(\emptyset)\}$.

Theorem 2. *If T is superstable, trivial or one-based, and if $U(T) \geq \omega^\omega$ then $I(T, \aleph_0) = 2^{\aleph_0}$.*

Proof. Suppose T is trivial or one-based and $U(T) \geq \omega^\omega$. For each positive integer n choose a type p_n over a finite domain such that $U(p_n) = \omega^n$. Each p_n is ESN by Proposition 3.1; hence the family $\{p_n | n \in \omega\}$ satisfies the assumptions of Proposition 5.1 and $I(T, \aleph_0) = 2^{\aleph_0}$. □

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